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Given a combinatorial game, can we determine if there exists a strategy for a player to win the game, and can we pinpoint what this strategy is? The answer to these questions varies from game to game, and even the most trivial games can become a burden to solve if we change a few rules, such as playing the game under the misère play rule. In this paper, we learn some fundamental techniques that are useful to solving many games. We will analyze the game of Nim and its many variations, and learn about the Sprague-Grundy function and how to create a single game out of many. Using the techniques we learned, we analyze and completely solve the Green Hackenbush game.

COMBINATORIAL GAME THEORY

by

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CHAPTER I

INTRODUCTION

The formal definition of a combinatorial game is quite delicate, as there are certain conditions that need to be satisfied, and there are a few things that are not allowed:

Definition 1. (Combinatorial Game) *A **combinatorial game** is a game that satisfies the following conditions:*

- *There are two players.*
- *There is a set of possible positions in the game. This set is usually finite.*
- *The rules specify all the possible moves between two different positions, for both players.*
- *The players alternate moves. A player cannot pass on making a move if it is his or her turn to move.*
- *The game ends when a player reaches a position from which no further moves are possible for the player whose turn it is to move.*
- *If a game never ends, it is declared a draw. However, most games have the **ending condition**, which requires that the game always ends in a finite number of moves.*

A **winning position** for a player in a combinatorial game is a position that guarantees the victory for that player. A **terminal position** is a position from

which no other moves are possible, and therefore, the game ends when one of the player reaches such a position.

Combinatorial games are divided into two categories: impartial and partisan games. In impartial games, the winning positions and the set of legal moves between positions is exactly the same for both players. Every other game is classified as a partisan game. Tic-tac-toe, chess, and checkers are all partisan games, since each player has a different set of moves available from a typical position. As evidenced by these examples, some partisan games can result in a tie, where neither player wins decisively. Under the **normal play** rule, a player that reaches a terminal position first is declared the winner of the game. Under **misère** rule, the first player to reach a terminal position loses. Unless we explicitly mention it, all games will be played according to normal play.

Our main goal in the study of a given combinatorial game is to discover whether either of the players has a specific strategy to reach a terminal position, therefore forcing the win, under normal play. Under misère play, we are trying to find a set of moves that forces the opponent to reach the terminal position. If such a set of moves exists, we call it the winning strategy for that player, and we would like to discover what this strategy is.

All of the previous definitions are by Ferguson [2].

Now that we are familiar with the concept of combinatorial games, it is time to introduce a very basic example of such a game, and attempt to a winning strategy, if one exists. For clarity, when we write Player I or the first player, we are referring to the player that makes the first move in the given combinatorial game. Player II or the second player is the player that responds to the first player's move.

Example 1. *Consider a pile of 7 coins, and two players alternating moves, removing one, two, or three coins from the pile at a time. The player that removes the*

last coin wins. Find a winning strategy for either of the players, if it exists.

When we analyze combinatorial games, backwards induction is often the best approach. We find all terminal positions, and then analyze the positions that led to these terminal positions and then the positions that lead to these and so on. In our example, there is only one terminal position (a pile of 0 coins). There are three positions from which we could have made a move to reach this terminal position: a pile with 1, 2, 3 coins, respectively. Then, from a pile of 4 coins, any move results in one of these three positions. So, if we are starting with 7 coins, the first player can remove 3 coins to obtain a pile of 4 coins, and then the second player is forced to move to a position with 1, 2 or 3 coins. The first player then can simply move to the terminal position and win.

Later in the paper, we will formalize these kinds of arguments into powerful tools for the analysis of combinatorial games, the characteristic properties of positions and the Sprague-Grundy Function. We will then use these tools to analyze several different types of games. While we are not able to fully solve a game, in many games we can at least determine which player can win given appropriate moves.

CHAPTER II

GRAPH GAMES

There exists a strong link between combinatorial games and graph theory. To further analyze this relation, we need to familiarize ourselves with the concept of directed graphs. Unless specified, the definitions and theorems in this chapter are by Ferguson [2].

Definition 2. A **directed graph** G is a pair (X, F) where X is a non-empty set of vertices and F is a set of ordered pairs of vertices, called directed edges.

For our purposes, the set X denotes all possible positions in a given game, and set F contains all possible moves from one position to another one. For a position $x \in X$, $F(x) = \{y \in X : (y, x) \in F\}$ is the set of all possible moves from position x . $F(x)$ is called the set of **followers of x** . If a position x has no followers (in other words, if $F(x)$ is empty), it is called a **terminal** position.

Let us consider a two player combinatorial game. Player 1 starts at position $x_0 \in X$, and players alternate moves. At a position x , a player whose turn it is to move moves to a position y so that $y \in F(x)$, or a follower of x . The player that is unable to move loses. We will focus on **progressively bounded graphs**, in which every directed path is of finite length, to avoid the situation where our game could continue indefinitely. Games corresponding to these graphs are also called progressively bounded, meaning they satisfy the ending condition. If X is finite, this means that the graph contains no directed cycles.

Recall the simple subtraction game we used in Example 1 in previous section. Our subtraction set was $S = \{1, 2, 3\}$, and our seven chips represent vertices, so

$X = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Our terminal position is the empty pile, so $F(0) = \emptyset$, $F(1) = 0$, $F(2) = \{0, 1\}$ and in general, $F(k) = \{k-3, k-2, k-1\}$ if $k \geq 3$. See Figure 2.1 for a depiction of this game as a graph.

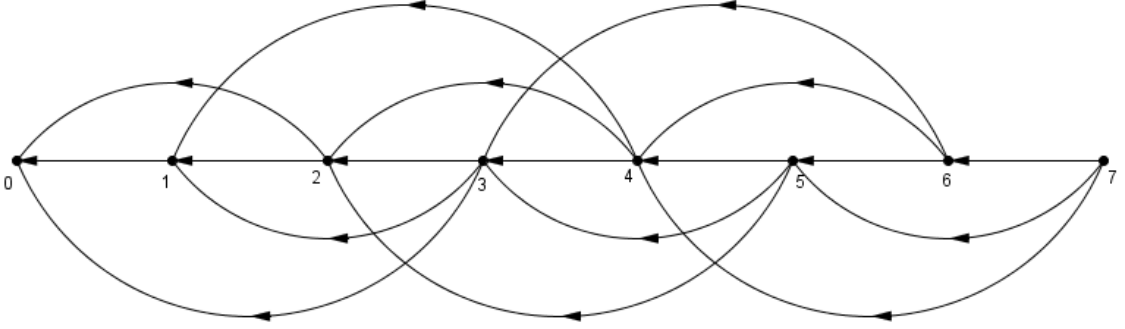


Figure 2.1: What Example 1 looks like as a graph.

From this simple example, we can see that positions with 1, 2, 3, 5, 6, or 7 coins guarantee the win for the first player to move from the starting position, and the position with 4 coins guarantees the win for the second player. We can see that these two sets of positions determine which player, first or second, has a winning strategy. We now wish to analyze these two types of different positions in more detail.

2.1 P and N positions

Definition 3. (P-positions, N-Positions) *A P-position is a position that guarantees the win for the second player to move from that position (i.e., we can think of this as the previous player), and an N-position is a position that guarantees the win for the player whose turn it is to move (i.e., the next player). The set of all P positions is denoted by \mathcal{P} . Similarly, the set of all N positions is denoted by \mathcal{N} .*

For brevity, we may refer to a P-position as P, or N-position as N.

Notice that if the entire set of positions is $\mathcal{P} \cup \mathcal{N} = X$, then one of the players must have a winning strategy. If the starting position is in \mathcal{N} , the first player to move has a winning strategy. Otherwise, the second player can win given using appropriate moves. Also, $\mathcal{P} \cap \mathcal{N} = \emptyset$ because for a position to be P and N guarantees the win for both players, which cannot happen. Also, if x is a position in a progressively bounded game, then $x \in \mathcal{P} \cup \mathcal{N}$. Perez [6] shows why.

Theorem 1. *In a progressively bounded impartial combinatorial game, all positions x lie in $\mathcal{P} \cup \mathcal{N}$.*

Proof. Since our game is bounded, it must end in a finite number of moves from any position. Let $b(x)$ denote the maximum number of moves that the game might last, from a position x . Using induction on $b(x)$, we show that every position is either \mathcal{P} or \mathcal{N} . If $b(x) = 0$, x is a terminal position, so $x \in \mathcal{P}$. Assume the theorem holds true for all positions x with $b(x) \leq n$. Let B denote the set of all those positions. Consider a position y so that $b(y) = n + 1$. Then any move from y will take us to a position in B . We may assume $B \subseteq \mathcal{P} \cup \mathcal{N}$ by induction.

If $F(y) \subseteq \mathcal{N}$, then $y \in \mathcal{P}$. Otherwise, there exists $z \in B$, a follower of y , such that $z \in \mathcal{P}$. Then, by definition, $y \in \mathcal{N}$.

This tells us that every position in a progressively bounded game is either a P-position or an N-position. □

For a given progressively bounded game, it is possible to label each position as either N or P by working recursively from the terminal positions. We now give the algorithm for normal play:

1. Label every terminal positions as a P-position.

2. Label every position that can reach an already labeled P-position in exactly one move as an N-position.
3. Find all positions whose every move leads to an already labeled N-position, and label them as P.
4. If no new positions are found in the previous step, we are done. Otherwise, repeat the process from step 2.

The algorithm for misère play is the same except we label the terminal positions as P-positions, and steps 2 and 3 are switched.

In general, when deciding whether a given position is P or N, we need to check the following three properties:

Characteristic Property. *The following three statements define P and N-positions recursively:*

1. *All terminal positions are P-positions.*
2. *From every N-position, there is at least one move to a P-position.*
3. *From any P-position, every move leads to an N-position*

All of these properties were proven in the previous theorem.

We are now ready to introduce a well-studied impartial combinatorial game, Nim.

CHAPTER III

THE GAME OF NIM

3.1 Overview

The Game of Nim is the perfect starting point for studying impartial games, as all progressively bounded games under normal play are simply Nim in disguise, as we shall later see. The game of Nim is played as follows:

There are several piles of chips, n in all. Let x_1, x_2, \dots, x_n be the number of chips in piles $1, 2, \dots, n$ respectively. The game is played between two players. A move consists of player choosing a pile, and removing at least one chip from that pile. The player may also choose to remove the whole pile. Whichever player removes the last chip overall is declared the winner.

3.2 Analysis of Nim

One-pile Nim does not make a very interesting game, since the first player can simply remove the whole pile and win. Nim with two piles is slightly more complicated. Assuming the numbers of chips in each pile were unequal to begin with, the winning strategy for the first player is to maintain the same number of chips in each pile. Consequently, the second player is always forced to make a move to a position where piles have a different number of chips. Since the terminal position, $(x_1, x_2) = (0, 0)$ contains the same number of chips, the second player cannot reach the terminal position. Therefore, the first player wins with this strategy. If the number of chips in each pile is the same in the beginning, the first player is forced to make a move

to some position with an uneven number in each pile. The second player can now use the same strategy as described to win.

3.3 Nim-Sum

The binary numeral system is used heavily in determining the winning strategy for Nim. For every integer $n \geq 0$, there exists a unique sequence $\epsilon_0, \epsilon_1, \epsilon_2, \dots$ so that $\epsilon_i = 0$ or 1 , and $n = \sum_{i=0}^{\infty} \epsilon_i 2^i$. We write this as $n = (\epsilon_m \epsilon_{m-1} \dots \epsilon_1 \epsilon_0)$. Normal binary addition is performed in the same manner as the regular addition, with carry. However, for our purposes, we need a slightly different type of sum, defined by Ferguson [2]:

Definition 4. The *nim-sum* of $(x_m \dots x_0)_2$ and $(y_m \dots y_0)_2$ is the binary number $(z_m \dots z_0)_2$, denoted by $(x_m \dots x_0)_2 \oplus (y_m \dots y_0)_2 = (z_m \dots z_0)_2$, where for all k , $z_k = x_k + y_k \pmod{2}$.

For our purposes, when performing calculations, we will drop the parentheses and the subscript 2 in the binary notation. So, for a position (x_1, x_2, \dots, x_n) in Nim, each x_i is the number of chips written in binary form $x_{ik}x_{i(k-1)} \dots x_{i1}x_{i0}$ where x_{ij} is 0 or 1.

3.4 The Winning Strategies for Nim

The winning strategy for the game of Nim is best expressed in terms of Nim-sum, and Ferguson [2] shows the solution:

Theorem 2. (Bouton's Theorem) *A position (x_1, x_2, \dots, x_n) in Nim is a P-position if and only if $x_1 \oplus x_2 \oplus \dots \oplus x_n = 0$.*

Proof. Let Z be the set of all positions whose Nim-sum is zero, and let NZ denote its complementary set, all positions whose Nim-sum is not zero. To prove the theorem,

we need to check that Z and NZ satisfy the three characteristic properties of P and N -positions: all terminal positions are Z , every move from any position in Z leads to a position in NZ , and there exists at least one move from any position in NZ to a position in Z .

The only terminal position is $t = (0, 0, \dots, 0)$, its Nim-sum is 0 and $t \in Z$.

Now we wish to show that for any given position in Z , every move leads to a position in NZ . Suppose $(x_1, x_2, \dots, x_n) \in Z$. Then, $x_1 \oplus x_2 \oplus \dots \oplus x_n = 0$. Without loss of generality, assume a move is made in the first pile. We obtain a new pile, y_1 , with $y_1 < x_1$. If we assume the new position, (y_1, x_2, \dots, x_n) , is in Z , then $y_1 \oplus x_2 \oplus \dots \oplus x_n = 0$. And so,

$$y_1 \oplus x_2 \oplus \dots \oplus x_n = 0 = x_1 \oplus x_2 \oplus \dots \oplus x_n.$$

Notice that for Nim-sums $x \oplus x = 0$, so the cancellation law holds, $y \oplus x = z \oplus x$ implies $y = z$. By canceling x_2, \dots, x_n from both sides, we obtain

$$x_1 = y_1$$

which is a contradiction. Therefore, $(y_1, x_2, \dots, x_n) \in NZ$.

The only thing that's left to check is the existence of a move from every position in NZ to a position in Z . We construct such a move in the following manner:

For a position $(x_1, x_2, \dots, x_n) \in NZ$, let $z = x_1 \oplus \dots \oplus x_n \neq 0$. There is at least one i such that $z_i = 1$. Find the largest i so that $z_i = 1$, and pick a pile x_j that has a 1 in the i th entry. By taking away an appropriate number of chips from that pile, we can change x_{ji} from 1 to 0, and also change the digits $x_{j(i-1)}, x_{j(i-2)}, \dots, x_{j0}$ to be what ever we want. Thus we can set $z_{i-1} = z_{i-2} = \dots = z_0 = 0$ as well. We need to remove exactly $x_j - (z \oplus x_j)$ chips from the j^{th} pile. This leaves the j^{th} pile with $z \oplus x_j$ chips, and the Nim-sum of the piles becomes 0. This is a legal move

because $b = z \oplus x_j$ has all bits $b_l = 0$ for $l \geq i$. Thus $b \leq 2^i - 1 < 2^i \leq x_j$, i.e. $b < x_j$.

Since Z and NZ possess the same characteristic properties as P and N -positions, $Z = \mathcal{P}$ and $NZ = \mathcal{N}$.

□

3.5 Misère Nim

Under misère play rule, the terminal position $(0, 0, \dots, 0)$ is the only N -terminal position, and therefore all positions $(1, 0, \dots, 0)$, $(0, 1, 0, \dots, 0), \dots, (0, 0, \dots, 0, 1)$ are P -positions. Let us assume that all of our piles are of size 1. We can observe that positions that have an odd number of piles with 1 chip are in \mathcal{P} , and the ones with an even number are in \mathcal{N} . Bouton determined a winning strategy for Nim under misère play based on reducing the game to an odd number of piles of size 1. Ferguson [2] explains:

Assuming the starting position has a non-zero Nim-sum, the first player can win. The game is split into two states. The first state of the game is when there are two or more piles with more than one chip. The first player can win by playing the game as if it were regular Nim, i.e., P -positions are those whose Nim-sum is 0 as long as there are two or more piles with more than one chip.

When the opponent makes a move and reduces the game to exactly one pile with more than one chip, we are in the second state of the game. This is guaranteed to happen because optimal play in ordinary Nim never requires the first player to leave a single pile of size greater than 1, since the Nim-sum resulting from playing the winning strategy is always 0. Also, the opponent cannot make a move from two or more piles of size greater than 1 to none. The P -position for the first player is obtained by making the move in the large pile, reducing it to zero or one chip,

whichever leaves odd number of piles of size 1 in play. Then the second player is forced to remove the last chip.

In general, the Misère version of combinatorial games is much more complicated to analyze and solve, even if the game under normal play is trivial.

3.6 Moore's Nim

In 1910, E. H. Moore [5] invented Nim_k , a generalization of Nim. As in Nim, there are n chips divided into piles, however, we may now remove any number of chips from each of a set of up to k piles. Therefore, Nim_1 is the ordinary game of Nim. To solve Nim_k , we define a sum analogous to Nim-Sum:

Definition 5. (Nim_k -Sum) *For a position (x_1, x_2, \dots, x_n) in Moore's Nim, the Nim_k -sum, denoted by $\oplus_k(x_1, x_2, \dots, x_n)$, is a number expressed as $y_m \dots y_0$ where $y_i \equiv x_{1i} + x_{2i} + \dots + x_{ni} \bmod (k+1)$ and $0 \leq y_i \leq k$ for all i .*

The definition and the following theorem are due to Moore [5], as shown by Peres [6].

Theorem 3. (Moore's Theorem) *A position (x_1, x_2, \dots, x_n) is a P-position if and only if its Nim_k -sum is 0. Therefore, a position is an N-position if and only if its Nim_k -sum is not 0.*

To better understand the theorem and the following proof, it helps to view this addition in terms of rows and columns. Each pile written in binary corresponds to one row, and for each $j, m \geq j \geq 0$, column j corresponds to a set consisting of $x_{1j}, x_{2j}, \dots, x_{nj}$:

$$x_1 = x_{1m}x_{1(m-1)} \dots x_{10}$$

$$x_2 = x_{2m}x_{2(m-1)} \dots x_{20}$$

$$x_n = x_{nm}x_{n(m-1)} \cdots x_{n0}$$

Our theorem tells us that a position is a P-position if the sum of entries in each column of the binary representation of addition is divisible by $k + 1$.

Proof. To prove the theorem, it is sufficient to show that our candidates for P-positions and N-positions satisfy the three characteristic properties:

1. The only terminal position is $(0, 0, \dots, 0)$, and it is a P-position since the sum of piles is zero.
2. It is possible to construct a move from a position whose Nim_k sum is not zero to a position whose Nim_k sum is zero in the following manner:

Consider the left-most column whose sum s is not divisible by $k + 1$. Let $s \equiv r \pmod{k + 1}$, where $r \in \{1, 2, \dots, k\}$. Select r rows whose entry in that column is 1. Changing these entries to 0 constitutes a legal move in a game of Nim, and for that column, the new sum is $s^* \equiv 0 \pmod{k + 1}$. For the remainder of this process, we are able to adjust any x_{ij} entry to the right of this column in the selected rows as we desire, since this still qualifies as a legal move. We proceed to the new left-most column whose sum $s_1 \not\equiv 0 \pmod{k + 1}$, ignoring the rows already selected. Let $q \equiv s_1 + r \pmod{k + 1}$, $q \in \{0, 1, \dots, k\}$.

This leads to two cases. If $q \leq r$, set the x_{ij} in q rows of the already selected rows to 0, and those in the other $r - q$ rows to 1. Then the new sum in this column is divisible by $k + 1$. If $r < q$, then we can select an additional $q - r$ of the previously non-selected rows that have a 1 in this column, and then changing the entries of all selected rows to 0 in this column gives us a sum that is divisible by $k + 1$.

We proceed with the step above if necessary until all columns have a Nim_k -

sum of 0 or until we select k rows. When we select k rows, the sum in any of the remaining columns will be between 0 and k , disregarding the selected rows. If the sum in a column is $0 \bmod (k + 1)$, set every entry in the k selected rows to 0. Otherwise, we have enough free rows to make each sum a multiple of $k + 1$.

3. Assume the game is currently in a position x where the Nim_k sum of all piles is 0. Given an arbitrary move, consider the left-most column where a change has occurred in the binary expansion. In x , the sum of entries in this column was divisible by $k + 1$. In this column, in order for a move to be legal, some 1's had to be changed to zeros, otherwise we would be *increasing* the number of chips. Since we are removing chips from up to k piles, and 1's are changed into zeros in this given column, the sum will decrease by at least 1, and at most k , and the new sum cannot be divisible by $k + 1$. Therefore, every move from a position whose Nim_k sum is zero leads to a position whose Nim_k is not zero.

□

It is interesting to note that for misère Nim_k , the winning strategy is a simple extension of the strategy for misère Nim. Recall that in misère Nim the strategy involved playing the winning strategy under the regular rules as long as there are at least 2 piles whose size is greater than 1.

Theorem 4. *The first player has a winning strategy in misère Nim_k precisely when the Nim_k -sum of the starting position is not zero. When there are $k + 1$ or more piles with more than 1 chip, P -positions are those whose Nim_k -sum is 0. When there are k or less piles with more than 1 chip, we reach a P -position by reducing all of those piles to either 0 or 1 in order to obtain a Nim_k -sum of 1.*

Proof. The winning strategy for the first player is as follows:

As long as there are at least $k + 1$ piles whose size is greater than 1, play as if the game were regular Nim_k . That is, the P-positions are those whose Nim_k -sum is 0, if there are more than $k + 1$ piles with more than one chip. When the opponent makes a move so that there are $n \leq k$ piles of size greater than 1, and $r \geq 0$ piles of size 1, the first player reduces all n piles to 0 or 1, whichever yields a Nim_k sum of 1 when summed with those r piles. In other words, when the second player makes a move so that there are $k + 1$ or less piles with more than one chip, P-positions are those whose Nim_k -sum is 1 after reducing all those piles to 0 or 1 chip.

This is guaranteed to happen because the first player cannot make a move to a position with k or less piles each of size greater than 1, since such positions don't have a Nim_k sum of 0. Therefore, the second player is forced to make such a move and furthermore is forced to leave a number of piles n , with $1 \leq n \leq k$. When the first player reduces all those piles to 0 or 1 so that the Nim_k -sum is 1, the game essentially becomes a misère take away game with k chips, and the second player is moving from a P-position. Therefore, the first player has a winning strategy. \square

3.7 Wythoff's Nim

Wythoff [8] came up with a Nim-like game, Wythoff's Nim which is played on only two piles of sizes m and n respectively. Peres [6] investigates it. Legal moves are the same as those of Nim, with an additional option to remove the same number of chips from both piles. Thus, our legal moves consist of reducing m to some value between 0 and $m - 1$ without changing n , reducing n to some value between 0 and $n - 1$ without changing m , or reducing both n and m by the same amount. Note that this immediately implies that $(k, k) \notin \mathcal{P}$, for $k > 0$ ($(0, 0)$ is the terminal position).

Before we analyze this game, consider another game played on a chessboard of arbitrary size n by m . A queen piece is positioned in the upper right corner of the board. Two players alternate moving the queen down, left, or diagonally towards the bottom left corner. The player who reaches the bottom left corner wins.

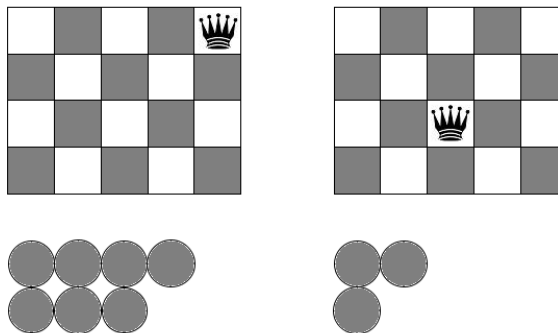


Figure 3.1: Corner the Queen, same as Wythoff's Nim. For more details, see Peres [6].

Let us denote the position of the queen by (x, y) , where $0 \leq x \leq n$ and $0 \leq y \leq m$. Considering legal moves of the queen, we see that this game is simply a version of Wythoff's Nim! Using backwards induction, consider the board of infinite size and let us analyze some of the P-positions. The bottom left corner is our terminal position. Any blocks to the right, up, or diagonal to it are N-positions:

Continuing in this fashion and crossing out the N-positions, we obtain the Figure 3.3.

There are a couple of things we must notice. Our graph is symmetrical along the main diagonal. In Wythoff Nim, position (n, m) is equivalent to (m, n) , so in this generalized version, the two mirror images are also equivalent. There can only be one P-position in each column and row. Otherwise, we would violate the characteristic property of P-position. Also, every column has to contain a P-position. Otherwise,

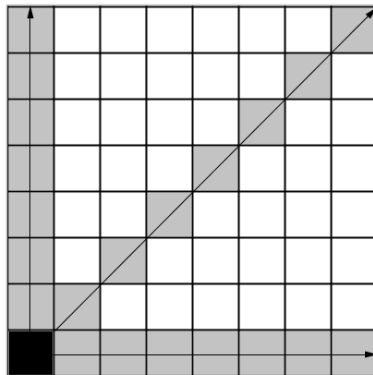


Figure 3.2: Any position that can reach $(0,0)$ is an N-position.

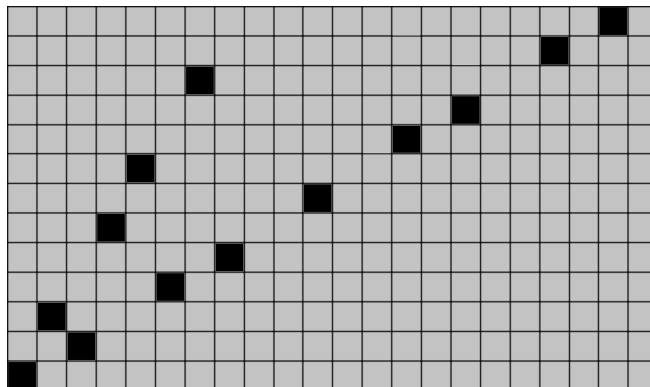


Figure 3.3: Some of the first P-positions obtained starting from $(0,0)$.

if a column k didn't have a P-position, all entries in that column would be N-positions and there should be a move that takes us to a P-position from each. Since we only have up to $k - 1$ columns to move to, that means that not every N-position in our k column can move to a P-position, a contradiction. Also, by symmetry, every row contains exactly one P-position as well. This yields another important fact. Let $A_0 = B_0 = 0$, and for $k \geq 1$ let (A_k, B_k) be the elements of \mathcal{P} with k^{th}

smallest x-coordinate among those above the line $y = x$. We have

$$\mathcal{P} = \{(0, 0)\} \cup \{(A_k, B_k) : k \geq 1\} \cup \{(B_k, A_k) : k \geq 1\}$$

and $\{A_k : k \geq 1\}$ and $\{B_k : k \geq 1\}$ form a partition of $\mathbb{N}^* = \{1, 2, \dots\}$.

| | | | | | | | | | | | |
|-------|---|---|---|---|----|----|----|----|----|----|-----|
| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | ... |
| A_k | 0 | 1 | 3 | 4 | 6 | 8 | 9 | 11 | 12 | 14 | ... |
| B_k | 0 | 2 | 5 | 7 | 10 | 13 | 15 | 18 | 20 | 23 | ... |

Notice that both A_k and B_k are increasing sequences, and $B_k > A_k$ for $k \geq 1$.

It turns out that these two sequences can be defined recursively. First, we need to know what the **minimal excludant** of a set is, as defined by Conway [1]:

Definition 6. *For a set S of non-negative integers, the minimal excludant or mex of S is the smallest non-negative integer n so that $n \notin S$.*

For example, if $S = \{0, 2, 4, 6\}$, then $\text{mex}(S) = 1$, since 1 is the smallest integer not in this set.

We define two sequences: for $k \geq 0$, let

$$a_k = \text{mex}\{a_0, \dots, a_{k-1}, b_0, \dots, b_{k-1},\}$$

$$b_k = a_k + k.$$

Note that for $k = 0$, $a_0 = \text{mex}\{\emptyset\} = 0$ and $b_0 = a_0 + 0 = 0$.

We now show that $A_k = a_k$ and $B_k = b_k$ for all $k \geq 0$. It's easy to check the following two theorems due to Wythoff [8]:

Theorem 5. $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ form a partition of \mathbb{N}^* .

Proof. We show by induction on j that $\{a_i\}_{i=1}^j$ and $\{b_i\}_{i=1}^j$ are two disjoint strictly increasing subsets of \mathbb{N}^* . This is true when $j = 0$, since both sets are empty.

Suppose $\{a_i\}_{i=1}^{j-1}$ and $\{b_i\}_{i=1}^{j-1}$ are disjoint and strictly increasing. From the way the a_i are defined, $a_j, a_{j-1} \notin \{a_0, a_1, \dots, a_{j-2}, b_0, b_1, \dots, b_{j-2}\}$, but a_{j-1} is the mex of that set so $a_{j-1} < a_j$. Since $a_j \notin \{b_0, \dots, b_{j-1}\}$, $\{a_i\}_{i=1}^j$ and $\{b_i\}_{i=1}^{j-1}$ are disjoint. Also, for each $i < j$,

$$b_j = a_j + j > a_i + j > a_i + i = b_i > a_i$$

and $\{b_i\}_{i=1}^j$ is also disjoint from $\{a_i\}_{i=1}^j$, and is increasing.

We now show that $\{a_i\}_{i=1}^j \cup \{b_i\}_{i=1}^j$ includes every integer in $\{1, \dots, j\}$. This is true for $j = 0$. If it is true for j , then $j + 1$ is either in $\{a_i\}_{i=1}^j \cup \{b_i\}_{i=1}^j$, or it is excluded, in which case $a_{j+1} = j + 1$, by the definition of a_{j+1} .

Therefore, the theorem holds. \square

Theorem 6. \mathcal{P} for Wythoff's Nim is precisely $W = \{(a_k, b_k) : k = 0, 1, 2, \dots\} \cup \{(b_k, a_k) : k = 0, 1, 2, \dots\}$.

Proof. We need to check the three characteristic properties of P-positions.

The only terminal position is $(0, 0)$, which is in W .

Consider $(m, n) = (a_k, b_k) \in W$. By definition, $b_k - a_k = k$, so piles m and n differ in k chips. If we reduce both piles by some amount, the difference between them is still k , and the only position in W with that difference is (a_k, b_k) , from the way the sequences are defined. If we reduce m , n only occurs with m in W , and similarly, if we reduce n , m occurs only with n in W . So any move from W leads to a position not in W .

Let $(m, n) \notin W$ with $m \leq n$, and let $k = n - m$. If $m > a_k$ and $n > b_k$, we can reduce both piles to (a_k, b_k) since a_k and b_k differ in k chips. Otherwise, either $m = a_j$ or $m = b_j$ for some $j < k$. If $m = a_j$, we can remove $k - j$ chips from n to get $(a_j, b_j) \in W$. If $m = b_j$ for some j , then $n \geq m = b_j > a_j$, so we can reduce n to a_j and go to position $(b_j, a_j) \in W$.

Therefore, $W = \mathcal{P}$. □

Now we wish to show that there exists a quick way to check if a given position is in \mathcal{P} .

Theorem 7. For $k \geq 0$, $a_k = \lfloor \frac{k(1+\sqrt{5})}{2} \rfloor$ and $b_k = \lfloor \frac{k(3+\sqrt{5})}{2} \rfloor$, where $\lfloor x \rfloor$ denotes the greatest integer n so that $n \leq x$.

Proof. Let $\alpha_k(\theta) = \lfloor \frac{k}{\theta} \rfloor$ and $\beta_k(\theta) = \lfloor \frac{k}{1-\theta} \rfloor$, for some fixed irrational $\theta \in (0, 1)$.

Our goal is to show that $\{\alpha_k(\theta)\}_{k=1}^{\infty}$ and $\{\beta_k(\theta)\}_{k=1}^{\infty}$ form a partition of \mathbb{N}^* , and then find θ so that $\alpha_k(\theta) = a_k$ and $\beta_k(\theta) = b_k$.

For any k , $\alpha_k \leq \alpha_{k+1}$, and similarly, $\beta_k \leq \beta_{k+1}$. Now, $\alpha_k(\theta) = N$ if and only if

$$N \leq \frac{k}{\theta} < N + 1,$$

$$\theta N \leq k < \theta N + \theta.$$

Also, $\beta_l(\theta) = N$ if and only if

$$N \leq \frac{l}{1-\theta} < N + 1,$$

$$\theta N + \theta - 1 < -l + N \leq \theta N.$$

There is exactly one integer m in the interval $(\theta N + \theta - 1, \theta N + \theta)$. Since θ is irrational, $m \neq N\theta$. Thus either m is in $(\theta N, \theta N + \theta)$ and $\alpha_k = N$, or m is in $(\theta N + \theta - 1, \theta N)$, and $\beta_l = N$. Therefore, $\{\alpha_k(\theta)\}_{k=1}^{\infty}$ and $\{\beta_k(\theta)\}_{k=1}^{\infty}$ are disjoint and form a partition of \mathbb{N}^* .

Now we want to find a θ so that

$$\alpha_k(\theta) = a_k, \quad \beta_k(\theta) = b_k.$$

Since $b_k = a_k + k$, we are looking for a solution to

$$\left\lfloor \frac{k}{\theta} \right\rfloor + k = \left\lfloor \frac{k}{1-\theta} \right\rfloor$$

Then,

$$\frac{1}{k} \left\lfloor \frac{k}{\theta} \right\rfloor + 1 = \frac{1}{k} \left\lfloor \frac{k}{1-\theta} \right\rfloor$$

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \left\lfloor \frac{k}{\theta} \right\rfloor + 1 = \lim_{k \rightarrow +\infty} \frac{1}{k} \left\lfloor \frac{k}{1-\theta} \right\rfloor$$

$$\frac{1}{\theta} + 1 = \frac{1}{1-\theta}$$

Therefore, $\theta^2 + \theta - 1 = 0$, and $\theta = \frac{\sqrt{5}-1}{2} = \frac{2}{1+\sqrt{5}} \in (0, 1)$, while the other root is negative. This number θ is also known as Φ , or the inverse of the golden ratio. Let us consider $\alpha_k(\theta)$ and $\beta_k(\theta)$ for $\theta = \frac{2}{1+\sqrt{5}}$. Since Φ satisfies

$$\frac{1}{\Phi} + 1 = \frac{1}{1-\Phi}$$

then

$$\frac{k}{\Phi} + k = \frac{k}{1-\Phi}$$

.

Since k is an integer, taking floor of both sides we obtain

$$\left\lfloor \frac{k}{1-\Phi} \right\rfloor = \left\lfloor \frac{k}{\Phi} \right\rfloor + k$$

so $\beta_k = \alpha_k + k$.

The only property left to check is

$$\alpha_k = \text{mex}\{\alpha_0, \alpha_1, \dots, \alpha_{k-1}, \beta_0, \beta_1, \dots, \beta_{k-1}\}.$$

We know that α_k can't be any of the values in the set of excludants. It remains to show that it is in fact the mex of all these values. By way of contradiction, suppose $\alpha_k \neq m = \text{mex}\{\alpha_0, \alpha_1, \dots, \alpha_{k-1}, \beta_0, \beta_1, \dots, \beta_{k-1}\}$. Then $m < \alpha_k \leq \alpha_l \leq \beta_l$ for all $k \leq l$. For $i \in \{0, 1, \dots, k-1\}$, $m \neq \alpha_i, \beta_i$, and since α_k, β_k span \mathbb{N}^* , m is missed. Therefore, $\alpha_k = \text{mex}\{\alpha_0, \alpha_1, \dots, \alpha_{k-1}, \beta_0, \beta_1, \dots, \beta_{k-1}\}$.

So, $a_k = \lfloor \frac{k(1+\sqrt{5})}{2} \rfloor$ and $b_k = \lfloor \frac{k(3+\sqrt{5})}{2} \rfloor$. □

Wythoff [8] himself said that he came up with this discovery “out of a hat”.

CHAPTER IV

THE SPRAGUE-GRUNDY FUNCTION

4.1 The Sprague-Grundy Function

Now that we know a combinatorial game can be represented by a directed graph, we need tools to analyze the game from this new perspective.

Definition 7. *The **Sprague-Grundy function** of a progressively bounded graph (X, F) is a function g defined on X , so that for $x \in X$,*

$$g(x) = \text{mex}\{g(y) \mid y \in F(x)\}.$$

Perez [6] defines the Sprague-Grundy function (SG for short) recursively, and as such, it is appropriate to start analyzing SG function of a graph starting with terminal positions. Since a terminal position t has no followers, $F(t) = \emptyset$, and their SG value is therefore $g(t) = 0$. All positions that only lead to a terminal position therefore have SG value of 1.

As an example, consider Figure 4.1.

There are four terminal positions in this graph, so we label them 0. Any positions that lead to one of the terminal positions are labeled 1. Continuing in this manner and using the definition of mex, we obtain the picture in Figure 4.2.

From the way P-positions and N-positions are defined, it is easy to see that for positions x for which $g(x) = 0$, $g \in \mathcal{P}$. This is easily checked by making sure that the three characteristic properties hold. Per Ferguson [2]:

1. All terminal positions $t \in \mathcal{P}$, since $g(t) = 0$.

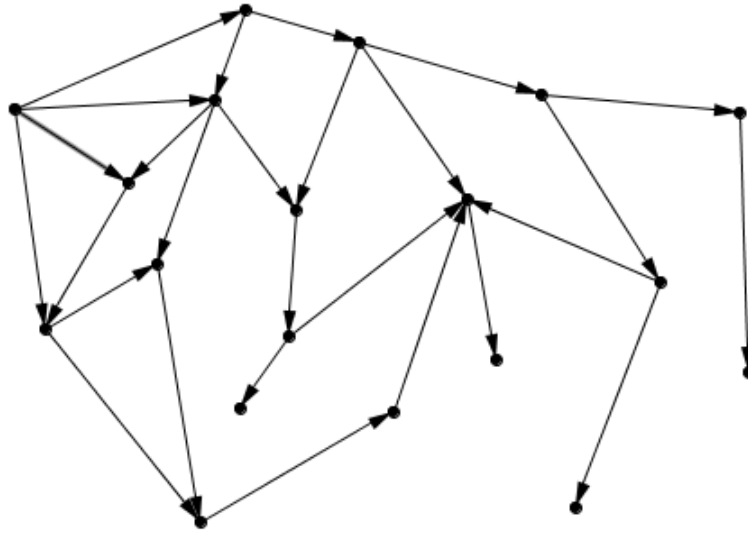


Figure 4.1: A graph game with no SG-values assigned to positions.

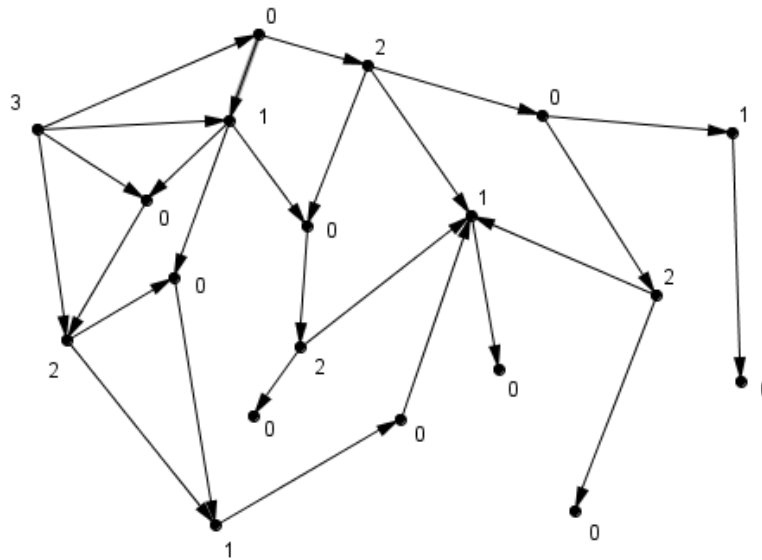


Figure 4.2: Graph of Figure 4.1 labeled with appropriate SG-values.

2. For a position x , let $g(x) = 0$. Assume there exists a follower $y \in F(x)$ so that $g(y) = 0$. But this cannot happen, since in that case x itself cannot have SG value of 0, from our definition of mex. So every follower of x has SG value different from 0.
3. For a position x so that $g(x) \neq 0$, there must exist at least one follower y so that $g(y) = 0$. If this was not true, then the SG-value of x itself would have to be 0, from the way mex is defined.

4.2 Sums of Games

For any collection of combinatorial games, we can create a new game that contains all of them. This game is played just like any other combinatorial game. Players alternate moves, and a move consists of a player picking a game and performing a legal move, as defined for that game. The first player unable to make a move loses.

Definition 8. (The Sum of Graph Games) *Let $G_1 = (X_1, F_1)$ and $G_2 = (X_2, F_2)$ be two progressively bounded graphs. $G = (X, F)$ is the sum of G_1 and G_2 , denoted by $G = G_1 + G_2$ if $X = X_1 \times X_2$, and for each $(x_1, x_2) \in X$, the set of followers is:*

$$F(x_1, x_2) = (F(x_1) \times \{x_2\}) \cup (\{x_1\} \times F(x_2)).$$

Naturally, we can sum more than two games in the same manner.

For example, a Nim game with 2 or more piles is simply a sum of several single-pile Nim games. While a single Nim pile is a very basic game, a three pile Nim is not quite so simple. This is true in general sums. Also, since each G_i is progressively bounded, the sum itself is progressively bounded, and the total number of moves is the sum of number of moves in each G_i .

Now that we are familiar with the sum of games, we can define what it means for two games to be equivalent. The definition is due to Perez [6].

Definition 9. Consider any two arbitrary combinatorial games G_1 and G_2 , with positions x_1 and x_2 , respectively. For some arbitrary combinatorial game H with position h , consider two sum games $G_1 + H$ and $G_2 + H$. If the outcome of (x_1, h) in $G_1 + H$ is the same as the outcome in (x_2, h) in $G_2 + H$, then G_1 and G_2 are equivalent.

The following theorem (due to Perez [6]) shows that any progressively bounded combinatorial game is equivalent to some Nim pile. Note that the SG-value of any single Nim pile of size k is just k , since the followers of a k -chip pile are $k - 1, k - 2, \dots, 1$, and 0 .

Theorem 8. Let G be a progressively bounded impartial combinatorial game under normal rules, with starting position x . Then G is equivalent to a Nim pile of size $g(x)$.

Proof. Let G be a progressively bounded graph for a combinatorial game whose SG-value is n . Let N be a Nim pile of size n , and let A be any arbitrary graph for a game. Consider the two games, $G + A$ and $N + A$. Assume player Q has a winning strategy for $N + A$. This can be either the first or the second player. If we can show this player also has a winning strategy for $G + A$, this shows that the two games G and N are equivalent.

For every move in $N + A$, we can perform a corresponding move in $G + A$. If a move is made in the A portion of the game, the other player responds with the same move in $G + A$. If a player makes a move in the N part of $N + A$, thus taking its SG-value to some $m < n$ then the next player can respond by making a move in G and taking its SG-value to m . Notice that m has to be smaller than n , since removing something from a Nim pile lowers its SG-value. Also, from the way the Sprague-Grundy function is defined, there must exist a move in G to make it's

SG-value m . Notice that not every move in $G + A$ has a corresponding move in $N + A$. It is possible to make a move in G so that the new SG-value is greater than x , since a follower of a position does not have to have a smaller SG-value. If such a move is made in G , it is impossible to mimic that move in N .

Player Q can use the exact same strategy as used for $N + A$, with one minor addition to compensate for the moves in $G + A$ that don't have corresponding move in $N + A$. If the other player makes a move in G that takes its SG-value to $m > n$, then player Q can perform a move in G to bring the SG-value back to n . This is possible since n has to be a follower of m , from the way SG-values are defined. These two moves cancel each other out, and since G is progressively bounded, only a limited number of such moves can be made, and player Q can always respond to them to bring the game to the original state in terms of SG-values.

Therefore, if the first or second player has a winning strategy for $G + A$, that player also has a winning strategy for $G + N$, and G and N are equivalent. \square

For a sum of games where we know the SG-value of each individual game, it is only natural to ask ourselves if there's a way to calculate the SG-value of this sum. The following is due to Sprague [7] and Grundy [3].

Theorem 9. (The Sprague-Grundy Theorem) *If g_i is the Sprague-Grundy function of G_i , $i = 1, \dots, n$, then $G = G_1 + \dots + G_n$ has Sprague-Grundy function $g(x_1, \dots, x_n) = g_1(x_1) \oplus g_1(x_2) \oplus \dots \oplus g_n(x_n)$.*

Proof. Let $x = (x_1, \dots, x_n)$ be an arbitrary point of X . Let b denote the Nim sum of all Sprague-Grundy numbers of each x_i , $b = g_1(x_1) \oplus \dots \oplus g_n(x_n)$. We need to show that no follower of x has SG-value of b , and that for every non-negative integer $a < b$, there exists a follower of x that has SG-value of a .

To show that no follower of x has the same SG-value, let us assume that

the opposite holds. So x has a follower of the same SG-value. Without loss of generality, supposed a move is made in the first game, so (x'_1, x_2, \dots, x_n) is a follower of (x_1, x_2, \dots, x_n) satisfying $g_1(x'_1) \oplus g_2(x_2) \oplus \dots \oplus g_n(x_n) = g_1(x_1) \oplus g_2(x_2) \oplus \dots \oplus g_n(x_n)$. Since the cancellation law holds in modular arithmetic, $g_1(x'_1) = g_1(x_1)$. Since no position can have a follower of the same SG-value, this is a contradiction.

Now we need to show that for $a < b$, x has a follower whose g-value is b . Let $d = a \oplus b$, and let k denote the number of digits in the binary expansion of d . Then, d has a 1 in the k^{th} position from the right since the left most digit is always 1 in a binary expansion of a number. From our assumption, $a < b$ so b has a 1 and a has a 0 in the k^{th} position, otherwise their Nim sum would not be d . Since $b = g_1(x_1) \oplus \dots \oplus g_n(x_n)$, there is at least one x_i whose binary expansion of $g_i(x_i)$ contains a 1 in k^{th} position. Without loss of generality, assume $i = 1$. Consider $d \oplus g_1(x_1)$, which has a 0 in the k^{th} position. Then $d \oplus g_1(x_1) < g_1(x_1)$. Therefore, there exists a move from x_1 to some x'_1 with $g_1(x'_1) = d \oplus g_1(x_1)$. Also, the move from (x_1, \dots, x_n) to (x'_1, \dots, x_n) is a legal move in G , with

$$g_1(x'_1) \oplus g_2(x_2) \oplus \dots \oplus g_n(x_n) = d \oplus g_1(x_1) \oplus g_2(x_2) \oplus \dots \oplus g_n(x_n) = d \oplus b$$

Observe that $d \oplus b = a \oplus b \oplus b = a$.

Therefore, the Sprague-Grundy theorem holds. □

CHAPTER V

GREEN HACKENBUSH

A rooted graph is an undirected graph where every vertex is connected to one of a set of specific nodes called the root or the ground. In a game of Hackenbush, players take turns deleting an edge from the rooted graph, and then deleting the components that are no longer connected to any vertex connected to the ground. When a player deletes an edge, we will say that he chopped the edge. There are several versions of this game. The impartial version is Green Hackenbush in which both players may chop any edge they wish. The edges are considered to be all colored green in Green Hackenbush. The partisan version is called Blue-Red Hackenbush, consisting of red and blue colored edges. In this version, one player may only chop blue edges, and the other may only chop red ones. The general game of Hackenbush combines these two versions: one player removes blue edges, other removes red edges, and either player can remove green edges. More information about those can be found in *On Numbers and Games* by Conway [1].

Lets consider the simplest example of Green Hackenbush, bamboo stalks:

A bamboo stalk is a finite length path, with no other edges attached to any vertices, with one end-vertex being attached to the ground or some other vertex in the graph, and the other end-vertex is a pendant vertex. Our game consists of a finite set of bamboo stalks. Thus a move consist of selecting an edge, and removing it and all the edges above. The player that removes the last segment of the group of bamboo stalks wins. This is just Nim, where chips are represented by segments of the bamboo stalk. Therefore, we already know the winning strategy for this game!

If we replace bamboo stalks with a collection of trees (a tree is a connected

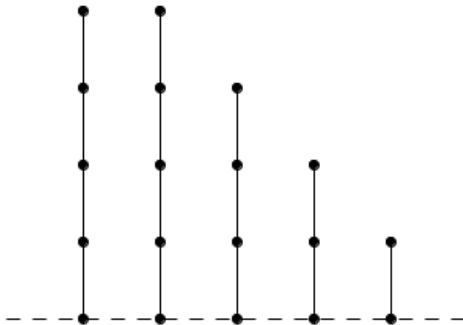


Figure 5.1: Bamboo Stalks.

graph with no cycles), where each tree has exactly one vertex in the ground, we have a slightly more complicated problem. Nonetheless, equipped with the right tool, the problem is once again reduced to Nim.

Theorem 10. (The Colon Principle) *Consider a fixed arbitrary graph G , and a vertex x in that graph, and any other two tree graphs H_1 and H_2 with the same Sprague-Grundy value. Let $G_1 = G : H_1$ and $G_2 = G : H_2$ be obtained from G by attaching H_1 and H_2 to x , respectively. Then $G_1 + G_2$ has a Sprague-Grundy value of 0, and G_1 and G_2 have the same SG-value.*

Proof. The theorem is by Guy [4], and the proof is shown by Ferguson [2].

For our claim to hold, it suffices to show that the second player has a winning strategy for $G_1 + G_2$. Consider all the moves the first player can make. If the first player makes a move in the G component of either graph, the second player can respond with the symmetric move. In case x is deleted after the first player makes a move, then the symmetric move by the second player results in two identical graphs whose sum is zero. If no move deletes H_1 or H_2 , the first player can only make so many moves in the G component, and eventually, the move is made in either H_1 or

H_2 since our graph is progressively bounded. Without loss of generality, assume the first player makes a move in H_1 , obtaining some H'_1 with the SG-value of k . If k is smaller than the SG-value of H_1 , the second player moves in H_2 to a follower whose SG-value is k . If k is larger than the SG-value of H_1 , the second player moves in H_1 to a follower whose SG-value is equal to the original SG-value of H_1 (this move can be considered a “pause”, and only finitely many such “pauses” can occur).

Therefore, for every move the first player makes, the second player has a corresponding move, and since the game is progressively bounded, the second player will reach the terminal position.

□

The following is a direct result of the Colon Principle. A **branch** is a collection of stalks that share a common vertex.

Corollary 1. *When multiple stalks of a tree come together at a vertex, they can be replaced with a single stalk of length equal to the Nim-sum of stalks making up the branch.*

Let us see the Colon Principle in action. For example, consider the graph in Figure 5.2.

We start off by labeling the bamboo stalks with their SG-value (the length of the stalk) that are attached to the graph. When several bamboo stalks meet at a single vertex, we take their Nim-sum, and replace all of them by a bamboo stalk of that length. We repeat this process until we reduce graph down to a single bamboo stalk, and this stalk represents the SG-value of the whole graph. See Figure 5.3.

Another direct result of the Colon Principle is the Parity Principle.

Corollary 2. (The Parity Principle) *The nim-sum value of any sum of tree graphs has the same parity as the total number of edges.*

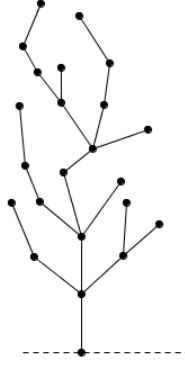


Figure 5.2: Tree Hackenbush.

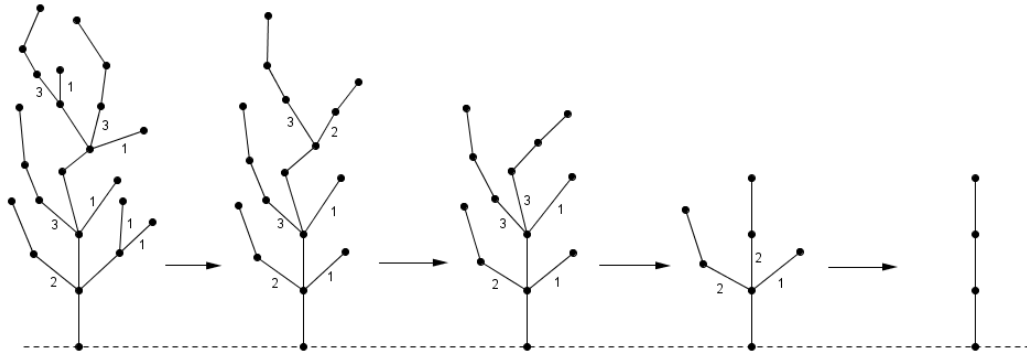


Figure 5.3: Applying the Colon Principle on a tree graph. All of these graphs are equivalent.

Proof. Notice we are performing two different types of addition. When we are calculating the SG-value of a stalk, we are performing ordinary addition, and when two or more stalks meet at a vertex, we perform Nim-addition. Since for a stalk, the SG-value is the same as the number of edges in the stalk, and since $m \oplus n$ is even if and only if $m + n$ is even, the number of edges directly determines the SG-value, and therefore the parity. \square

Let us turn to arbitrary graphs, which may contain cycles and loops, and

with multiple paths attached to the ground. Since any graph is equivalent to a Nim pile, our goal is to somehow find a tree that is equivalent to a given graph. This tree is then equivalent to a Nim pile, using the Colon principle.

In graph theory, by **identifying** vertices u and v in a graph we obtain a new graph where the two vertices are replaced by a single vertex w , with each edge between u and v replaced by a loop at w , and where edges that were incident on u or v are redirected to w . All other edges remain unchanged. We can apply this to a cycle in a graph: we simply contract all the edges by identifying any two or more vertices in the cycle by repeated application of identifying vertices. This process is called *fusing* by Conway and Guy [4]. For Green Hackenbush we can also replace any loops with a single edge, where one end is unattached. With repeated application of fusing and the Colon Principle, we can shrink any graph down to a single stalk. We now show that the stalk obtained by this procedure has the same SG-value as the original graph. The groundwork for this proof is due to Guy [4].

Theorem 11. (The Fusion Principle) *The vertices on any cycle may be fused without changing the Sprague-Grundy value of the graph.*

Proof. By way of contradiction, suppose the Fusion Principle doesn't hold. Among all counterexamples with the minimum possible number of edges, pick G with the smallest number of vertices.

Since we are assuming the Fusion Principle fails, G must exhibit certain properties:

1. All ground edges must meet in a single vertex. Since ground itself represents a vertex in our graphs, having multiple vertices touching the ground is redundant, and this condition is needed to meet our minimal counterexample requirements. See Figure 5.4.

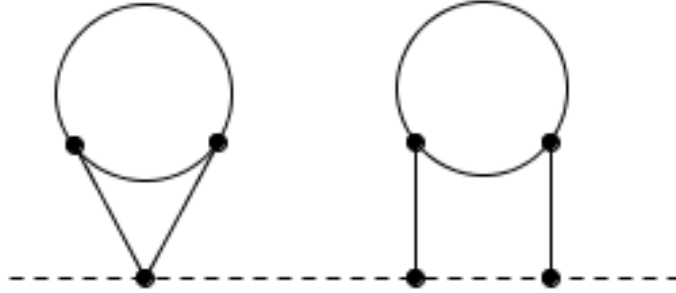


Figure 5.4: All ground nodes must meet in a single vertex. These two graphs are equivalent, but the one on the right contains an extra vertex which violates the way we defined the minimal counterexample.

2. For any pair of vertices, a and b , there cannot exist three or more edge-disjoint paths connecting them. See Figure 5.5.

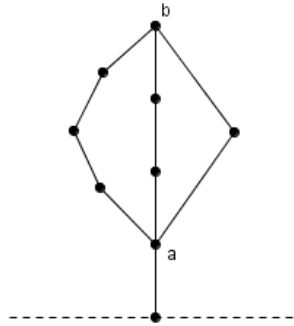


Figure 5.5: There cannot exist three or more distinct paths between any two vertices.

If there were, by fusing a and b we obtain a new graph H which contains n edges and $m - 1$ vertices, so the SG-value of H is different than G . Consider the sum of these two graphs. Then $G + H \neq 0$, and the first player can make a winning move. Suppose the first player makes a move in G , obtaining the new graph G' . The second player can make the corresponding move by symmetry,

removing the same edge in H , to obtain H' . The same is true if the first player makes a move in H to obtain H' , as the second player can respond by symmetry.

Now both G' and H' have at most $n - 1$ edges, and we can apply the Fusion Principle on each. But, since a and b were connected by three or more edge-disjoint paths in G , the first player could have removed at most one of those paths by removing an edge, so a and b are still on a cycle C in G' . Therefore, after fusing C in G' and the corresponding vertices, we obtain two identical graphs, and $G' + H' = 0$, which contradicts our assumption.

3. Every cycle in G has to include the ground node. See Figure 5.6.

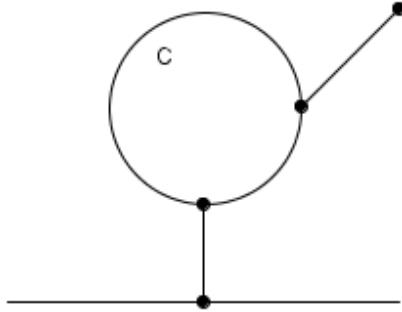


Figure 5.6: If there exists a cycle in our minimal counterexample, it must contain the ground node.

If G had some cycle C not touching the ground, and if G' is obtained by removing all edges in C , then G' contains only one vertex of C . Otherwise, if it contained two or more vertices, they would have to be connected by three disjoint paths, which we just showed cannot happen. So, C is connected to the rest of the graph at one distinct vertex, x . Notice that C itself could have some additional parts of the graph attached to it in vertices other than x . If we treat x as the ground node, we can now apply the Fusion Principle

on the subgraph that includes C obtaining a graph C' with the same SG-value. We can replace the original subgraph that contains C with C' and the value of the original graph remains unchanged. The new graph is no longer a counterexample.

4. G contains only one cycle that touches the ground. See Figure 5.7.

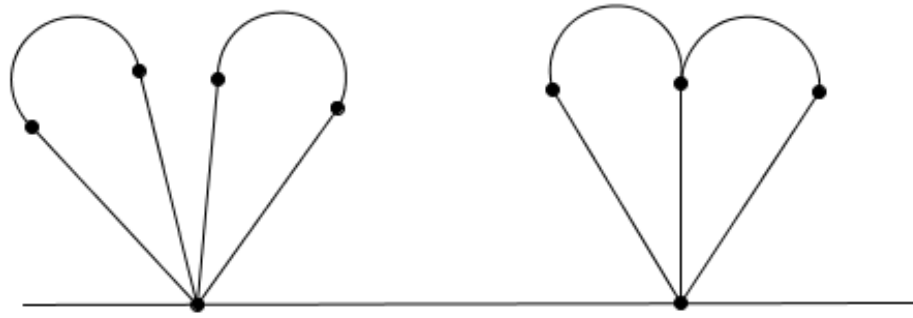


Figure 5.7: There cannot exist more than a single cycle touching the ground.

Otherwise, our graph would just be a sum of two or more cycles meeting at the ground node, on which we could apply the Fusion Principle separately, or they would have to be connected in some different way, but that violates our second property.

Combining properties 3 and 4, we see that G can only include one cycle. So our minimal counterexample has a cycle of k edges touching the ground, potentially with trees coming out of vertices not touching the ground, but we can apply the Colon Principle on trees to turn them into simple stalks. Refer to Figure 5.8 to see what the counterexample should look like. Let us look at what would happen if the Fusion Principle was applicable on this graph. Fusing the cycle would just yield k loops, each equivalent to a stalk with one edge, along with all the stalks that could

be attached to vertices in the cycle. If the SG-value of all the stalks is l , then the SG-value of the whole graph would be l if k was even, or $l + 1$ if k was odd.

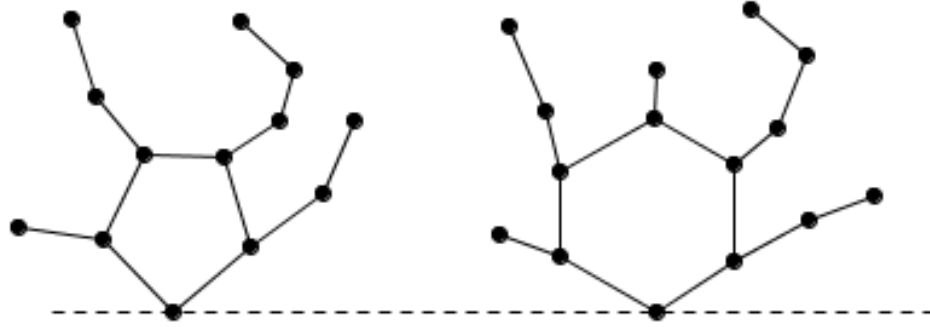


Figure 5.8: What minimal counterexample to the Fusion Principle should look like. The graph on the left contains odd number of edges in its cycle, and the graph on the right contains odd number of edges.

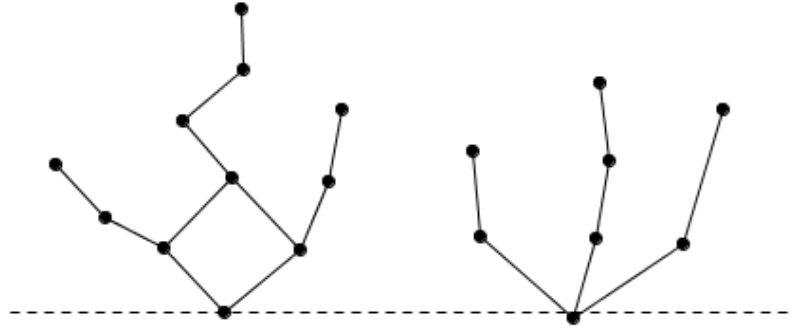


Figure 5.9: The sum game of minimal counterexample containing an even number of edges in its cycle and the graph obtained by applying the Fusion Principle.

So, if k was even, let us look at the sum of G and G' , where G' is obtained by fusing the cycle. See Figure 5.9. Since we assumed the Fusion Principle fails, $G' + G \neq 0$, and therefore, the first player has a winning strategy. Suppose the

first player makes a move in one of the stalks, in either G or G' . Then the second player can make a corresponding symmetrical move in the other graph. We can now apply the Fusion Principle on these and obtain identical graphs, and their sum is 0. Therefore, the first player has to make a move on the cycle. But, in doing so, we obtain two tree graphs. The number of edges left from the cycle is now odd, and the number of edges in all the stalks is even (since we have two copies), therefore the number of edges is odd. By the Parity Principle, the SG-value of this new graph is odd. Since the first player cannot make a move to a position whose SG-value is 0, the starting position, $G + G'$ must have value 0, and the Fusion Principle holds if k is even.

No move will take the first player to a P-position when k is odd. For the sake of brevity and due to the complexity of the proof in this case, we omit the steps needed to show the winning strategy for the second player. The reader may refer to *Winning Ways* (p.188-189) [4] for the existence of the winning strategy in this case, as well as the actual algorithm involved.

Therefore, the minimal counterexample G to the Fusion Principle does not exist.

□

For an alternate proof of the Fusion Principle using mating functions, see Conway [1].

We have now completely solved the Green Hackenbush game.

CHAPTER VI

CONCLUSION

We have analyzed Nim, Wythoff's Nim, Moore's Nim, Green Hackenbush, and the misère versions for some of these. The characteristic properties of P and N-positions and the Sprague-Grundy Function provided the groundwork that allows us to analyze these games. In some cases we could determine which player has a guaranteed win, even if we did not know what the winning strategy is.

The games we chose to study illustrated a number of different proof methods. In Wythoff's Nim, for example, we defined two sequences and then showed that they span the P-positions. Also, the Golden ratio ϕ miraculously came up in the discussion for generating the P-positions. In Green Hackenbush, we were able to establish a few tools to reduce complicated positions to regular Nim, which we can solve.

BIBLIOGRAPHY

- [1] Conway J. (1976) *On Numbers and Games*. Academic Press, London.
- [2] Ferguson T. (2008) *Game Theory*. University of California, Los Angeles. Web (http://www.math.ucla.edu/~tom/Game_Theory/Contents.html).
- [3] Grundy P. M., (1939). *Mathematics and games*. Eureka 2: 68. Reprinted, 1964, 27: 911.
- [4] Guy R., Conway J., and Berlekamp E. (1982) *Winning Ways for your mathematical plays, Volume I*. Academic Press, London.
- [5] Moore E. H., (1909) *A generalization of the game called nim*. Ann. of Math. (Ser. 2), 11:9394.
- [6] Peres Y., with contributions by Wilson D., (2009) *Game Theory, Alive*. University of California, Berkeley. Web. (<http://dbwilson.com/games>).
- [7] Sprague R. P., (1935). *Über mathematische Kampfspiele*. Tohoku Mathematical Journal 41: 438444.
- [8] Wythoff W. A., (1907) *A modification of the game of Nim*. Nieuw Arch. Wisk.,7:199202.